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Transient Anomaly Imaging in Visco-Elastic Media Obeying a Frequency Power-Law

Elie Bretin* Lili Guadarrama Bustos* Abdul Wahab*

May 31, 2010

Abstract

In this work, we consider the problem of reconstructing a small anomaly in a viscoelastic medium from wave-field measurements. We choose Szabo's model to describe the viscoelastic properties of the medium. Expressing the ideal elastic field without any viscous effect in terms of the measured field in a viscous medium, we generalize the imaging procedures, such as time reversal, Kirchhoff Imaging and Back propagation, for an ideal medium to detect an anomaly in a visco-elastic medium from wave-field measurements.

1 Introduction

We consider the problem of reconstructing a small anomaly in a viscoelastic medium from wave-field measurements. The Voigt model is a common model to describe the viscoelastic properties of tissues. Catheline *et al.* [10] have shown that this model is well adapted to describe the viscoelastic response of tissues to low-frequency excitations. We choose a more general model derived by Szabo *et al.* [16] that describes observed power-law behavior of many viscoelastic materials. It is based on a time-domain statement of causality [15]. It reduces to the Voigt model for the specific case of quadratic frequency loss. Expressing the ideal elastic field without any viscous effect in terms of the measured field in a viscous medium, we generalize the methods described in [2, 3, 4, 5, 8]; namely the time reversal, back-propagation and Krichhoff Imaging, to recover the viscoelastic and geometric properties of an anomaly from wave-field measurements.

The article is organized as follows. In section 2 we introduce a general visco-elastic wave equation. section 3 is devoted to the derivation of the Green function in a viscoelastic medium. In section 4 we present anomaly imaging procedures and reconstruction methods in visco-elastic media. Numerical illustrations are provided in section 5.

2 General Visco-Elastic Wave Equation

When a wave travels through a biological medium, its amplitude decreases with time due to attenuation. The attenuation coefficient for biological tissue may be approximated by a power-law over a wide range of frequencies. Measured attenuation coefficients of soft tissue typically have linear or greater than linear dependence on frequency [11, 15, 16].

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In an ideal medium; without attenuation, Hooke's law gives the following relationship between stress and strain tensors:

$$\mathcal{T} = \mathcal{C} : \mathcal{S} \tag{1}$$

where $\mathcal{T}$, $\mathcal{C}$ and $\mathcal{S}$ are respectively stress, stiffness and strain tensors of orders 2, 4 and 2 and $:$ represents tensorial product.

Consider a dissipative medium. Suppose that the medium is homogeneous and isotropic. We write

$$\mathcal{C} = [\mathcal{C}_{ijkl}] = [\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]\,, \tag{2}$$

$$\eta = [\eta_{ijkl}] = [\eta_s\delta_{ij}\delta_{kl} + \eta_p(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]\,, \tag{3}$$

where δ_{ab} is the Kronecker delta function, μ, λ are the Lamé parameters, and η_s, η_p are the shear and bulk viscosities, respectively. Here we have adopted the generalized summation convention over the repeated index.

Throughout this work we suppose that

$$\eta_p, \eta_s << 1. \tag{4}$$

For a medium obeying a power-law attenuation model and under the smallness condition (4), a generalized Hooke's law reads [16]

$$\mathcal{T}(x,t) = \mathcal{C} : \mathcal{S}(x,t) + \eta : \mathcal{M}(\mathcal{S})(x,t) \tag{5}$$

where the convolution operator $\mathcal{M}$ is given by

$$\mathcal{M}(\mathcal{S}) = \begin{cases} -(-1)^{y/2}\frac{\partial^{y-1}\mathcal{S}}{\partial t^{y-1}} & \text{y is an even integer,} \\ \frac{2}{\pi}(y-1)!(-1)^{(y+1)/2}\frac{H(t)}{t^y} * \mathcal{S} & \text{y is an odd integer,} \\ -\frac{2}{\pi}\Gamma(y)\sin(y\pi/2)\frac{H(t)}{|t|^y} * \mathcal{S} & \text{y is a non integer.} \end{cases} \tag{6}$$

Here $H(t)$ is the Heaviside function and Γ denotes the gamma function.

Note that for the common case, $y = 2$, the generalized Hooke's law (5) reduces to the Voigt model,

$$\mathcal{T} = \mathcal{C} : \mathcal{S} + \eta : \frac{\partial \mathcal{S}}{\partial t}. \tag{7}$$

Taking the divergence of (5) we get

$$\nabla \cdot \mathcal{T} = \left(\bar{\lambda} + \bar{\mu}\right)\nabla(\nabla\cdot\mathbf{u}) + \bar{\mu}\Delta\mathbf{u},$$

where

$$\bar{\lambda} = \lambda + \eta_p\mathcal{M}(\cdot) \quad \text{and} \quad \bar{\mu} = \mu + \eta_s\mathcal{M}(\cdot).$$

Next, considering the equation of motion for the system, *i.e.*,

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} - \mathbf{F} = \nabla\cdot\mathcal{T}, \tag{8}$$

with ρ being the constant density and $\mathbf{F}$ the applied force. Using the expression for $\nabla\cdot\mathcal{T}$, we obtain the generalized visco-elastic wave equation

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} - \mathbf{F} = \left(\bar{\lambda} + \bar{\mu}\right)\nabla(\nabla\cdot\mathbf{u}) + \bar{\mu}\Delta\mathbf{u}. \tag{9}$$

3 Green's Function

In this section we find the Green function of the viscoelastic wave equation (9). For doing so, we first need a Helmholtz decomposition.

3.1 Helmholtz Decomposition

The following lemma holds.

Lemma 3.1 *If the displacement field* $\mathbf{u}(x,t)$ *satisfies (9),* $\frac{\partial \mathbf{u}(x,0)}{\partial t} = \nabla A + \nabla \times B$ *and* $\mathbf{u}(x,0) = \nabla C + \nabla \times D$ *and if the body force* $\mathbf{F} = \nabla \varphi_f + \nabla \times \psi_f$ *then there exist potentials* φ_u *and* ψ_u *such that*

- $\mathbf{u} = \nabla \varphi_u + \nabla \times \psi_u$; $\nabla \cdot \psi_u = 0$;
- $\frac{\partial^2 \varphi_u}{\partial t^2} = \frac{\varphi_f}{\rho} + c_p^2 \Delta \varphi_u + \nu_p \mathcal{M}(\Delta \varphi_u) \approx \frac{\varphi_f}{\rho} - \frac{\nu_p \mathcal{M}(\varphi_f)}{\rho c_p^2} + c_p^2 \Delta \varphi_u + \frac{\nu_p}{c_p^2} \mathcal{M}(\partial_t^2 \varphi_u)$;
- $\frac{\partial^2 \psi_u}{\partial t^2} = \frac{\psi_f}{\rho} + c_s^2 \Delta \psi_u + \nu_s \mathcal{M}(\Delta \psi_u) \approx \frac{\psi_f}{\rho} - \frac{\nu_s \mathcal{M}(\psi_f)}{\rho c_s^2} + c_s^2 \Delta \psi_u + \frac{\nu_s}{c_s^2} \mathcal{M}(\partial_t^2 \psi_u)$,

with

$$c_p^2 = \frac{\lambda + 2\mu}{\rho}, \; c_s^2 = \frac{\mu}{\rho}, \; \nu_p = \frac{\eta_p + 2\eta_s}{\rho}, \quad \text{and} \quad \nu_s = \frac{\eta_s}{\rho}.$$

Proof. For φ_u and ψ_u defined as

$$\varphi_u(x,t) = \int_0^t \int_0^\tau \left[\frac{\varphi_f}{\rho} + (c_p^2 + \nu_p \mathcal{M})(\nabla \cdot u) \right] ds d\tau + tA + C \tag{10}$$

$$\psi_u(x,t) = \int_0^t \int_0^\tau \left[\frac{\vec{\psi}_f}{\rho} - (c_s^2 + \nu_s \mathcal{M})(\nabla \times u) \right] ds d\tau + t\vec{B} + \vec{D} \tag{11}$$

we have the required expression for $\mathbf{u}$. Moreover, it is evident from (11) that $\nabla \cdot \psi_u = 0$

Now, on differentiating φ_u and ψ_u twice with respect to time, we get

$$\frac{\partial^2 \varphi_u}{\partial t^2} = \frac{\varphi_f}{\rho} + c_p^2 \Delta \varphi_u + \nu_p \mathcal{M}(\Delta \varphi_u)$$

$$\frac{\partial^2 \psi_u}{\partial t^2} = \frac{\psi_f}{\rho} + c_s^2 \Delta \psi_u + \nu_s \mathcal{M}(\Delta \psi_u)$$

Finally, applying $\mathcal{M}$ on last two equations, neglecting the higher order terms in ν_s and ν_p and injecting back the expressions for $\mathcal{M}(\Delta \varphi_u)$ and $\mathcal{M}(\Delta \psi_u)$, we get the required differential equations for φ_u and ψ_u. $\square$

Let

$$K_m(\omega) = \omega \sqrt{\left(1 - \frac{\nu_m}{c_m^2} \hat{\mathcal{M}}(\omega)\right)}, \quad m = s, p, \tag{12}$$

where the multiplication operator $\hat{\mathcal{M}}(\omega)$ is the Fourier transform of the convolution operator $\mathcal{M}$.

If φ_u and ψ_u are causal then it implies the causality of the inverse Fourier transform of $K_m(\omega), m = s, p$. Applying the Kramers-Krönig relations, it follows that

$$-\Im m K_m(\omega) = \mathcal{H}\Big[\Re e K_m(\omega)\Big] \quad \text{and} \quad \Re e K_m(\omega) = \mathcal{H}\Big[\Im m K_m(\omega)\Big], \quad m = p, s, \tag{13}$$

where $\mathcal{H}$ is the Hilbert transform. Note that $\mathcal{H}^2 = -I$. The convolution operator $\mathcal{M}$ given by (6) is based on the constraint that causality imposes on (5). Under the smallness assumption (4), the expressions in (6) can be found from the Kramers-Krönig relations (13). One drawback of (13) is that the attenuation, $\Im m K_m(\omega)$, must be known at all frequencies to determine the dispersion, $\Re e K_m(\omega)$. However, bounds on the dispersion can be obtained from measurements of the attenuation over a finite frequency range [13].

3.2 Solution of (9) with a Concentrated Force.

Let u_{ij} denote the i-th component of the solution $\mathbf{u}_j$ of the elastic wave equation related to a force $\mathbf{F}$ concentrated in the x_j-direction. Let $j = 1$ for simplicity and suppose that

$$\mathbf{F} = -T(t)\delta(x-\xi)\mathbf{e}_1 = -T(t)\delta(x-\xi)(1,0,0), \tag{14}$$

where ξ is the source point and $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an orthonormal basis of $\mathbb{R}^3$. The corresponding Helmholtz decomposition of the force $\mathbf{F}$ can be written [14] as

$$\begin{cases} \mathbf{F} = \nabla \varphi_f + \nabla \times \psi_f, \\ \varphi_f = \frac{T(t)}{4\pi} \frac{\partial}{\partial x_1}\left(\frac{1}{r}\right), \\ \psi_f = -\frac{T(t)}{4\pi}\left(0, \frac{\partial}{\partial x_3}\left(\frac{1}{r}\right), -\frac{\partial}{\partial x_2}\left(\frac{1}{r}\right)\right), \end{cases} \tag{15}$$

where $r = |x-\xi|$.

Consider the Helmholtz decomposition for $\mathbf{u}_{i1}$ as

$$u_{i1} = \nabla \varphi_1 + \nabla \times \vec{\psi}_1 \tag{16}$$

where φ_1 and ψ_1 are the solutions of the equations

$$\Delta \varphi_1 - \frac{1}{c_p^2}\frac{\partial^2 \varphi_1}{\partial t^2} + \frac{\nu_p}{c_p^4}\mathcal{M}(\partial_t^2 \varphi_1) = \frac{\nu_p \mathcal{M}(\varphi_f)}{\rho c_p^4} - \frac{\varphi_f}{c_p^2 \rho}, \tag{17}$$

$$\Delta \psi_1 - \frac{1}{c_s^2}\frac{\partial^2 \psi_1}{\partial t^2} + \frac{\nu_s}{c_s^4}\mathcal{M}(\partial_t^2 \psi_1) = \frac{\nu_s \mathcal{M}(\psi_f)}{\rho c_s^4} - \frac{\psi_f}{c_s^2 \rho}. \tag{18}$$

Taking the Fourier transform of (16),(17) and (18) with respect to t we get

$$\hat{\mathbf{u}}_1 = \nabla \hat{\varphi}_1 + \nabla \times \hat{\psi}_1 \tag{19}$$

$$\Delta \hat{\varphi}_1 + \frac{K_p^2(\omega)}{c_p^2}\hat{\varphi}_1 = \frac{\nu_p \hat{\mathcal{M}}(\omega)\hat{\varphi}_f}{\rho c_p^4} - \frac{\hat{\varphi}_f}{\rho c_p^2}, \tag{20}$$

$$\Delta \hat{\psi}_1 + \frac{K_s^2(\omega)}{c_s^2}\hat{\psi}_1 = \frac{\nu_s \hat{\mathcal{M}}(\omega)\hat{\psi}_f}{\rho c_s^4} - \frac{\hat{\psi}_f}{\rho c_s^2}, \tag{21}$$

with $K_m(\omega), m = p, s$, given by (12).

It is well known that the Green's functions of the Helmholtz equations (20) and (21) are

$$\hat{g}^m(r,\omega) = \frac{e^{\sqrt{-1}\frac{K_m(\omega)}{c_m}r}}{4\pi r}, \quad m = s, p.$$

Thus, following [14] we write $\hat{\varphi}_1$ as

$$\hat{\varphi}_1(x,\omega;\xi) = -\left(1 - \frac{\nu_p \hat{\mathcal{M}}(\omega)}{c_p^2}\right)\frac{\hat{T}(\omega)}{\rho(4\pi c_p)^2}\int_V \hat{g}^p(x-\chi,\omega)\frac{\partial}{\partial \chi_1}\frac{1}{|\chi-\xi|}dV_\chi.$$

and divide the volume V into spherical shells of radius h centered at observation point x. On each shell $\hat{g}^p(x-\chi,\omega)$ rests constant. So we have

$$\hat{\varphi}_1(x,\omega;\xi) = -\left(1-\frac{\nu_p\hat{\mathcal{M}}(\omega)}{c_p^2}\right)\frac{\hat{T}(\omega)}{\rho(4\pi c_p)^2}\int_0^\infty \frac{1}{h}\hat{g}^p(h,\omega)\int_\sigma \frac{\partial}{\partial\chi_1}\frac{1}{R}d\sigma dh.$$

with $h=|x-\chi|$, $R=|\chi-\xi|$ and $d\sigma$ the appropriate surface element.

As [1]

$$\int_\sigma \frac{\partial}{\partial\chi_1}\left(\frac{1}{R}\right) = \begin{cases} 0 & h>r \\ 4\pi h^2 \frac{\partial}{\partial x_1}\left(\frac{1}{r}\right) & h<r \end{cases}$$

Therefore, we have following expression for $\hat{\varphi}_1$:

$$\hat{\varphi}_1(x,\omega;\xi) = -\left(1-\frac{\nu_p\hat{\mathcal{M}}(\omega)}{c_p^2}\right)\frac{\hat{T}(\omega)}{4\pi\rho}\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right)\int_0^{r/c_p}\zeta e^{\sqrt{-1}K_p(\omega)\zeta}\,d\zeta. \tag{22}$$

In the same way, the vector $\hat{\psi}_1$ is given by

$$\hat{\psi}_1(x,\omega;\xi) = \left(1-\frac{\nu_s\hat{\mathcal{M}}(\omega)}{c_s^2}\right)\frac{\hat{T}(\omega)}{4\pi\rho}\left(0,\frac{\partial}{\partial x_3}\left(\frac{1}{r}\right),-\frac{\partial}{\partial x_2}\left(\frac{1}{r}\right)\right)\int_0^{r/c_s}\zeta e^{\sqrt{-1}K_s(\omega)\zeta}\,d\zeta. \tag{23}$$

Introduce the following notation:

$$I_m(x,\omega) = A_m\int_0^{r/c_m}\zeta e^{\sqrt{-1}K_m(\omega)\zeta}\,d\zeta \tag{24}$$

$$E_m(x,\omega) = A_m e^{\sqrt{-1}K_m(\omega)\frac{r}{c_m}}, \tag{25}$$

$$A_m(\omega) = \left(1-\frac{\nu_m\hat{\mathcal{M}}(\omega)}{c_m^2}\right), \quad m=p,s. \tag{26}$$

We obtain, after a lengthy but simple calculation, that $\hat{u}_{i1}$ is given by

$$\hat{u}_{i1} = \frac{\hat{T}(\omega)}{4\pi\rho}\frac{\partial^2}{\partial x_i x_1}\left(\frac{1}{r}\right)[I_s(r,\omega)-I_p(r,\omega)] + \frac{\hat{T}(\omega)}{4\pi\rho c_p^2 r}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_1}E_p(r,\omega)$$
$$+\frac{\hat{T}(\omega)}{4\pi\rho c_s^2 r}\left(\delta_{i1}-\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_1}\right)E_s(r,\omega),$$

and therefore, it follows that the solution u_{ij} for an arbitrary j is

$$\hat{u}_{ij} = \frac{\hat{T}(\omega)}{4\pi\rho}(3\gamma_i\gamma_j-\delta_{ij})\frac{1}{r^3}[I_s(r,\omega)-I_p(r,\omega)] + \frac{\hat{T}(\omega)}{4\pi\rho c_p^2}\gamma_i\gamma_j\frac{1}{r}E_p(r,\omega)$$
$$+\frac{\hat{T}(\omega)}{4\pi\rho c_s^2}(\delta_{ij}-\gamma_i\gamma_j)\frac{1}{r}E_s(r,\omega),$$

where $\gamma_i = (x_i-\xi_i)/r$.

3.3 Green's function

If we substitute $T(t)=\delta(t)$, where delta is the Dirac mass, then the function $u_{ij} = G_{ij}$ is the i-th component of the Green function related to the force concentrated in the x_j-direction. In this case, we have $\hat{T}(\omega)=1$. Thus, we have the following expression for $\hat{G}_{ij}$:

$$\hat{G}_{ij} = \frac{1}{4\pi\rho}(3\gamma_i\gamma_j-\delta_{ij})\frac{1}{r^3}[I_s(r,\omega)-I_p(r,\omega)] + \frac{1}{4\pi\rho c_p^2}\gamma_i\gamma_j\frac{1}{r}E_p(r,\omega)$$
$$+\frac{1}{4\pi\rho c_s^2}(\delta_{ij}-\gamma_i\gamma_j)\frac{1}{r}E_s(r,\omega),$$

which implies that

$$\hat{G}_{ij}(r,\omega;\xi) = \hat{g}^p_{ij}(r,\omega) + \hat{g}^s_{ij}(r,\omega) + \hat{g}^{ps}_{ij}(r,\omega), \tag{27}$$

where

$$\hat{g}^{ps}_{ij}(r,\omega) = \frac{1}{4\pi\rho}\left(3\gamma_i\gamma_j - \delta_{ij}\right)\frac{1}{r^3}\left[I_s(r,\omega) - I_p(r,\omega)\right], \tag{28}$$

$$\hat{g}^p_{ij}(r,\omega) = \frac{A_p(\omega)}{\rho c_p^2}\gamma_i\gamma_j\hat{g}^p(r,\omega), \tag{29}$$

and

$$\hat{g}^s_{ij}(r,\omega) = \frac{A_s(\omega)}{\rho c_s^2}\left(\delta_{ij} - \gamma_i\gamma_j\right)\hat{g}^s(r,\omega). \tag{30}$$

Let $G(r,t;\xi) = (G_{ij}(r,t;\xi))$ denote the transient Green function of (9) associated with the source point ξ. Let $G^m(r,t;\xi)$ and $W_m(r,t)$ be the inverse Fourier transforms of $A_m(\omega)\hat{g}^m(r,\omega)$ and $I_m(r,\omega), m = p, s$, respectively. Then, from (27-30), we have

$$\begin{aligned} G_{ij}(r,t;\xi) &= \tfrac{1}{\rho c_p^2}\gamma_i\gamma_j G^p(r,t;\xi) + \tfrac{1}{\rho c_s^2}\left(\delta_{ij} - \gamma_i\gamma_j\right)G^s(r,t;\xi) \\ &\quad + \tfrac{1}{4\pi\rho}\left(3\gamma_i\gamma_j - \delta_{ij}\right)\tfrac{1}{r^3}\left[W_s(r,t) - W_p(r,t)\right]. \end{aligned} \tag{31}$$

Note that by a change of variables,

$$W_m(r,t) = \frac{4\pi}{c_m^2}\int_0^r \zeta^2 G^m(\zeta,t;\xi)\,d\zeta.$$

4 Imaging procedure

Consider the limiting case $\lambda \to +\infty$. The Green function for a quasi-incompressible visco-elastic medium is given by

$$\begin{aligned} G_{ij}(r,t;\xi) &= \tfrac{1}{\rho c_s^2}\left(\delta_{ij} - \gamma_i\gamma_j\right)G^s(r,t;\xi) \\ &\quad + \tfrac{1}{\rho c_s^2}\left(3\gamma_i\gamma_j - \delta_{ij}\right)\tfrac{1}{r^3}\textstyle\int_0^r \zeta^2 G^s(\zeta,t;\xi)\,d\zeta. \end{aligned}$$

To generalize the detection algorithms presented in [2, 3, 5, 4, 8] to the visco-elastic case we shall express the ideal Green function without any viscous effect in terms of the Green function in a viscous medium. From

$$G^s(r,t;\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\sqrt{-1}\omega t} A_s(\omega) g^s(r,\omega)\,d\omega,$$

it follows that

$$G^s(r,t;\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} A_s(\omega)\frac{e^{\sqrt{-1}(-\omega t + \frac{K_s(\omega)}{c_s}r)}}{4\pi r}\,d\omega.$$

4.1 Approximation of the Green Function

Introduce the operator

$$L\phi(t) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_0^{+\infty} A_s(\omega)\phi(\tau)e^{\sqrt{-1}K_s(\omega)\tau}e^{-\sqrt{-1}\omega t}\,d\tau\,d\omega,$$

for a causal function ϕ. We have

$$G^s(r,t;\xi) = L(\frac{\delta(\tau - r/c_s)}{4\pi r}),$$

and therefore,

$$L^* G^s(r,t;\xi) = L^* L(\frac{\delta(\tau - r/c_s)}{4\pi r}),$$

where L^* is the $L^2(0,+\infty)$-adjoint of L.

Consider for simplicity the Voigt model. Then, $\hat{\mathcal{M}}(\omega) = -\sqrt{-1}\omega$ and hence,

$$K_s(\omega) = \omega\sqrt{1 + \frac{\sqrt{-1}\nu_s}{c_s^2}\omega} \approx \omega + \frac{\sqrt{-1}\nu_s}{2c_s^2}\omega^2,$$

under the smallness assumption (4). The operator L can then be approximated by

$$\tilde{L}\phi(t) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_0^{+\infty} A_s(\omega)\phi(\tau) e^{-\frac{\nu_s}{2c_s^2}\omega^2\tau} e^{\sqrt{-1}\omega(\tau-t)}\, d\tau\, d\omega.$$

Since

$$\int_{\mathbb{R}} e^{-\frac{\nu_s}{2c_s^2}\omega^2\tau} e^{\sqrt{-1}\omega(\tau-t)}\, d\omega = \frac{\sqrt{2\pi}c_s}{\sqrt{\nu_s\tau}} e^{-\frac{c_s^2(\tau-t)^2}{2\nu_s\tau}},$$

and

$$\sqrt{-1}\int_{\mathbb{R}} \omega e^{-\frac{\nu_s}{2c_s^2}\omega^2\tau} e^{\sqrt{-1}\omega(\tau-t)}\, d\omega = -\frac{\sqrt{2\pi}c_s}{\sqrt{\nu_s\tau}}\frac{\partial}{\partial t} e^{-\frac{c_s^2(\tau-t)^2}{2\nu_s\tau}},$$

it follows that

$$\tilde{L}\phi(t) = \int_0^{+\infty} \frac{t}{\tau}\phi(\tau)\frac{c_s}{\sqrt{2\pi\nu_s\tau}} e^{-\frac{c_s^2(\tau-t)^2}{2\nu_s\tau}}\, d\tau. \tag{32}$$

Analogously,

$$\tilde{L}^*\phi(t) = \int_0^{+\infty} \frac{\tau}{t}\phi(\tau)\frac{c_s}{\sqrt{2\pi\nu_s t}} e^{-\frac{c_s^2(\tau-t)^2}{2\nu_s t}}\, d\tau. \tag{33}$$

Since the phase in (33) is quadratic and ν_s is small then by the stationary phase theorem A.1, we can prove the following theorem:

Theorem 4.1

$$\tilde{L}^*\phi \approx \phi + \frac{\nu_s}{2c_s^2}\partial_{tt}(t\phi), \quad \tilde{L}\phi \approx \phi + \frac{\nu_s}{2c_s^2} t\partial_{tt}\phi,$$

and

$$\tilde{L}^*\tilde{L}\phi \approx \phi + \frac{\nu_s}{c_s^2}\partial_t(t\partial_t\phi), \tag{34}$$

and therefore,

$$(L^*\tilde{L})^{-1}\phi \approx \phi - \frac{\nu_s}{c_s^2}\partial_t(t\partial_t\phi). \tag{35}$$

□

Proof. (See appendix A)

4.2 Reconstruction Methods

From the previous section, it follows that the ideal Green function, $\delta(\tau - r/c_s)/(4\pi r)$, can be approximately reconstructed from the viscous Green function, $G^s(r, t; \xi)$, by either solving the ODE

$$\phi + \frac{\nu_s}{c_s^2}\partial_t(t\partial_t\phi) = L^* G^s(r, t; \xi),$$

with $\phi = 0, t \ll 0$ or just making the approximation

$$\delta(\tau - r/c_s)/(4\pi r) \approx L^* G^s(r, t; \xi) - \frac{\nu_s}{c_s^2}\partial_t(t\partial_t L^* G^s(r, t; \xi)).$$

Once the ideal Green function $\delta(\tau - r/c_s)/(4\pi r)$ is reconstructed, one can find its source ξ using a time-reversal, a Kirchhoff or a back-propagation algorithm. See [2, 3, 4, 5].

Using the asymptotic formalism developed in [5, 6, 7], one can also find the shear modulus of the anomaly using the ideal near-field measurements which can be reconstructed from the near-field measurements in the viscous medium. The asymptotic formalism reduces the anomaly imaging problem to the detection of the location and the reconstruction of a certain polarizability tensor in the far-field and separates the scales in the near-field.

5 Numerical Illustrations

In this section, we illustrate the profile of the Green function. We choose parameters of simulation as in the work of Bercoff *et al.*[9]: we take $\rho = 1000$, $c_s = 1$, $c_p = 40$, $r = 0.015$ and $\nu_p = 0$.

In figure 1, we plot temporal representation of the green function:

$$t \to \frac{1}{\rho c_p^2}\left(G^p(r, t; \xi) + G^s(r, t; \xi)\right) + \frac{1}{4\pi\rho r^3}\left[W_s(r, t) - W_p(r, t)\right].$$

for three different values of y and ν_s. We can see that the attenuation behavior varies with respect to different choices of power law exponent y. One can clearly distinguish the three different terms of the Green function; *i.e.* G^s_{ij}, G^p_{ij} and G^{ps}_{ij}.

Figure 2 corresponds to spatial representation of the green function:

$$(x, y) \to \frac{1}{\rho c_p^2}\left((x/r)^2 G^p(r, t; \xi) + (1 - (x/r)^2) G^s(r, t; \xi)\right) + \frac{1}{4\pi\rho r^3}(3(x/r)^2 - 1)\left[W_s(r, t) - W_p(r, t)\right],$$

for different values of y at $t = 0.015$. As expected, we get a diffusion of the wavefront with the increasing values of y and depending the choice of ν_s.

In figure 3, we illustrate the results of the approximation of the operator $L\phi$ with the smooth function $\phi(t) = exp(-50 * (t - 1).^2)''$. As shown by the stationary phase theorem A.1 , the numerically calculated L^∞-error

$$\|L\phi - \left(\phi + \frac{\nu_s}{2c_s^2} t\phi''\right)\|_{L^\infty(\mathbb{R}^+)}$$

is of order two.

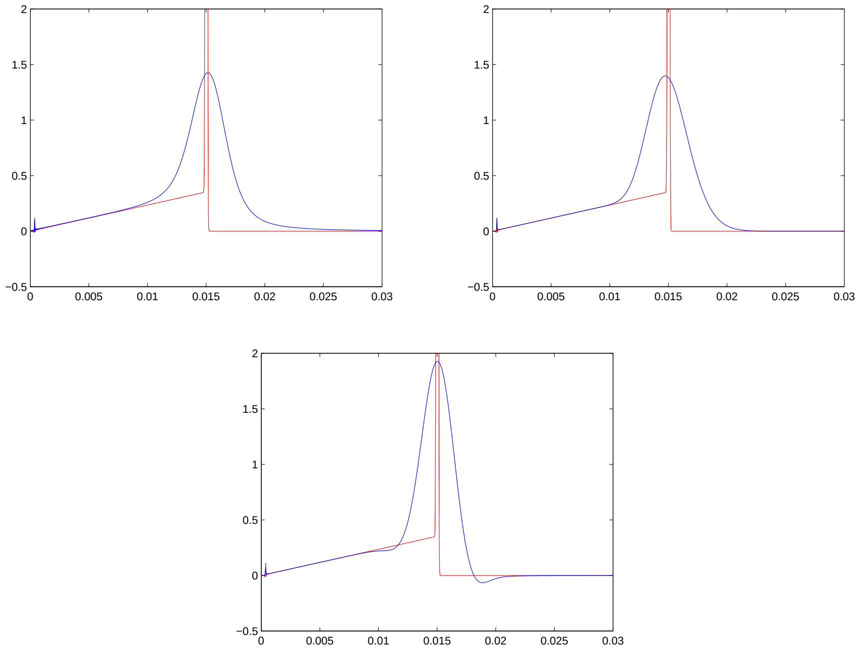

Figure 1: Temporal response to a spatio-temporal delta function using a purely elastic Green's function (red line) and a viscous Green's function (blue line): Left, $y = 1.5$, $\nu_s = 4$; Center, $y = 2$, $\nu_s = 0.2$; Right, $y = 2.5$, $\nu_s = 0.002$.

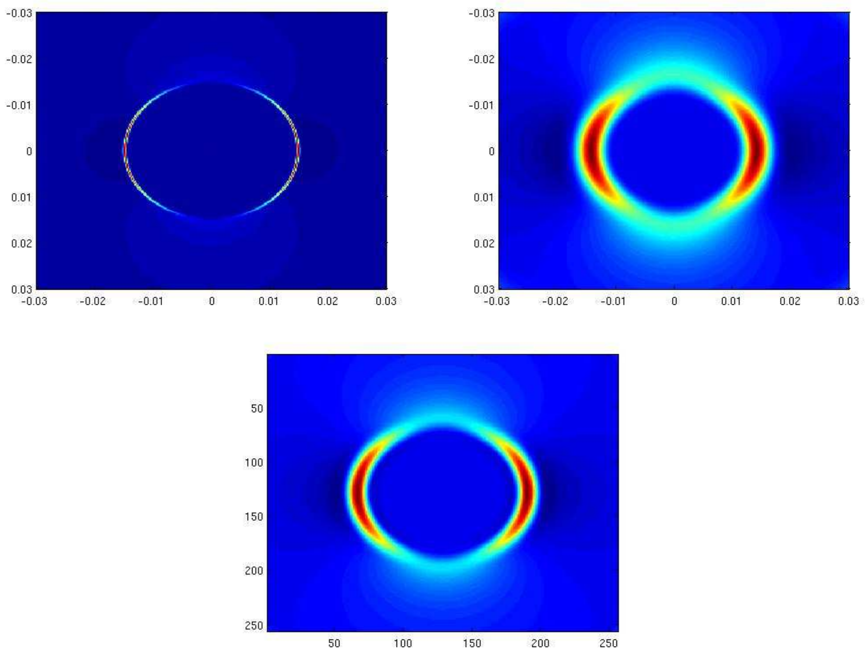

Figure 2: $2D$ spatial response to a spatio-temporal delta function at $t = 0.015$ with a purely elastic Green's function, a viscous Green's function with $y = 2$, $\nu_s = 0.2$ and $y = 2.5$, $\nu_s = 0.002$.

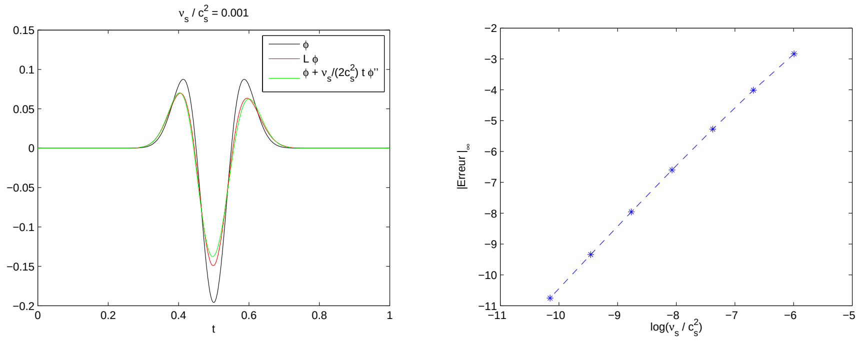

Figure 3: Approximation of L via stationary phase theorem : Left, comparison between $L\phi$ and $\phi + \frac{\nu_s}{c_s^2} t\phi''$ where $\frac{\nu_s}{c_s^2} = 0.0001$ and ϕ is a smooth function. Right: error $\frac{\nu_s}{c_s^2} \to \|L\phi - \phi + \frac{\nu_s}{c_s^2}\|_\infty$ in logarithmic scale.

6 Conclusion

In this paper, we have computed the Green function in a visco-elastic medium obeying a frequency power-law. For the Voigt model, which corresponds to a quadratic frequency loss, we have used the stationary phase theorem A.1 to reconstruct the ideal Green function from the viscous one by solving an ODE. Once the ideal Green function is reconstructed, one can find its source ξ using the algorithms in [2, 3, 4, 5] such as time reversal, back-propagation, and Kirchhoff Imaging. For more general power-law media, one can recover the ideal Green function from the viscous one by inverting a fractional derivative operator. This would be the subject of a forthcoming paper.

A Proof of Theorem (4.1)

The proof of theorem (4.1) is based on the following theorem (see [12, Theorem 7.7.1]).

Theorem A.1 (Stationary Phase) *Let $K \subset [0,\infty)$ be a compact set, X an open neighborhood of K and k a positive integer. If $\psi \in C_0^{2k}(K)$, $f \in C^{3k+1}(X)$ and $Im(f) \geq 0$ in X, $Im(f(t_0)) = 0$, $f'(t_0) = 0$, $f''(t_0) \neq 0$, $f' \neq 0$ in $K \setminus \{t_0\}$ then for $\epsilon > 0$*

$$\left| \int_K \psi(t) e^{if(t)/\epsilon} dx - e^{if(t_0)/\epsilon} \left(\lambda f''(t_0)/2\pi i\right)^{-1/2} \sum_{j<k} \epsilon^j L_j \psi \right| \leq C\epsilon^k \sum_{\alpha \leq 2k} \sup |\psi^{(\alpha)}(x)|.$$

Here C is bounded when f stays in a bounded set in $C^{3k+1}(X)$ and $|t - t_0|/|f'(t)|$ has a uniform bound. With,

$$g_{t_0}(t) = f(t) - f(t_0) - \frac{1}{2} f''(t_0)(t - t_0)^2,$$

which vanishes up to third order at t_0, we have

$$L_j \psi = \sum_{\nu - \mu = j} \sum_{2\nu \geq 3\mu} i^{-j} \frac{2^{-\nu}}{\nu! \mu!} (-1)^\nu f''(t_0)^{-\nu} (g_{t_0}^\mu \psi)^{(2\nu)}(t_0). \qquad \square$$

Note that L_1 can be expressed as the sum $L_1\psi = L_1^1\psi + L_1^2\psi + L_1^3\psi$, where L_1^j is respectively associate to the pair $(\nu_j, \mu_j) = (1,0), (2,1), (3,2)$ and is identified to

$$\begin{cases} L_1^1\psi &= \frac{-1}{2i} f''(t_0)^{-1}\psi^{(2)}(t_0), \\ L_1^2\psi &= \frac{1}{2^2 2! i} f''(t_0)^{-2}(g_{t_0}u)^{(4)}(t_0) = \frac{1}{8i} f''(t_0)^{-2}\left(g_{t_0}^{(4)}(t_0)\psi(t_0) + 4g_{t_0}^{(3)}(t_0)\psi'(t_0)\right), \\ L_1^3\psi &= \frac{-1}{2^3 2! 3! i} f''(t_0)^{-3}(g_{t_0}^2\psi)^{(6)}(t_0) = \frac{-1}{2^3 2! 3! i} f''(t_0)^{-3}(g_{t_0}^2)^{(6)}(t_0)\psi(t_0). \end{cases}$$

Now we turn to the proof of formula (34). Let us first consider the case of operator L^*. We have

$$\tilde{L}^*\phi(t) = \int_0^{+\infty} \frac{\tau}{t}\phi(\tau)\frac{c_s}{\sqrt{2\pi\nu_s t}} e^{-\frac{c_s^2(\tau-t)^2}{2\nu_s t}} d\tau = \frac{1}{t\sqrt{\epsilon}}\left(\int_0^{+\infty} \psi(\tau)e^{if(\tau)/\epsilon}\right),$$

with, $f(\tau) = i\pi(\tau - t)^2$, $\epsilon = \frac{2\pi\nu_s t}{c_s^2}$ and $\psi(\tau) = \tau\phi(\tau)$. Remark that the phase f satisfies at $\tau = t$, $f(t) = 0$, $f'(t) = 0$, $f''(t) = 2i\pi \neq 0$. Moreover, we have

$$\begin{cases} e^{if(t)/\epsilon}\left(\epsilon^{-1}f''(t)/2i\pi\right)^{-1/2} = \sqrt{\epsilon} \\ g_t(\tau) = f(\tau) - f(t) - \frac{1}{2}f''(t)(\tau-t)^2 = 0 \\ L_1\psi(t) = L_1^1\psi(t) = \frac{-1}{2i}f''(t)^{-1}\psi''(t) = \frac{1}{4\pi}(t\phi)''. \end{cases}$$

Thus, Theorem A.1 implies that

$$\left|\tilde{L}^*\phi(t) - \left(\phi(t) + \frac{\nu_s}{2c_s^2}(t\phi)''\right)\right| \leq \frac{C}{t}\epsilon^{3/2}\sum_{\alpha\leq 4}\sup|(t\phi)^{(\alpha)}|.$$

The case of the operator $\tilde{L}$ is very similar. Note that

$$\tilde{L}\phi(t) = \int_0^{+\infty} \frac{t}{\tau}\phi(\tau)\frac{c_s}{\sqrt{2\pi\nu_s\tau}} e^{-\frac{c_s^2(\tau-t)^2}{2\nu_s\tau}} d\tau = \frac{t}{\sqrt{\epsilon}}\left(\int_0^{+\infty} \psi(\tau)e^{if(\tau)/\epsilon}\right),$$

with $f(\tau) = i\pi\frac{(\tau-t)^2}{\tau}$, $\epsilon = \frac{\nu_s}{2\pi c_s^2}$ and $\psi(\tau) = \phi(\tau)\tau^{-\frac{3}{2}}$. It follows that

$$f'(\tau) = i\pi\left(1 - \frac{t^2}{\tau^2}\right), \quad f''(\tau) = 2i\pi\frac{t^2}{\tau^3}, \quad f''(t) = 2i\pi\frac{1}{t},$$

and the function $g_t(\tau)$ equals to

$$g_t(\tau) = i\pi\frac{(\tau-t)^2}{\tau} - i\pi\frac{(\tau-t)^2}{t} = i\pi\frac{(t-\tau)^3}{\tau t}.$$

We deduce that

$$\begin{cases} (g_t\psi)^{(4)}(t) &= \left(g_t^{(4)}(t)\psi(t) + 4g_t^{(3)}(t)\psi'(t)\right) = i\pi\left(\frac{24}{t^3}\psi(t) - \frac{24}{t^2}\psi'(t)\right) \\ (g_t^2\psi)^{(6)}(t) &= (g_t^2)^{(6)}(t)\psi(t) = -\pi^2\frac{6!}{t^4}\psi(t), \end{cases}$$

and then,

$$\begin{cases} L_1^1\psi = \frac{-1}{i}\left(\frac{1}{2}(f''(t))^{-1}\psi''(t)\right) = \frac{1}{4\pi}t\left(\frac{\tilde{\phi}}{\sqrt{t}}\right)'' = \frac{1}{4\pi}\left(\sqrt{t}\tilde{\phi}''(t) - \frac{\tilde{\phi}'(t)}{\sqrt{t}} + \frac{3}{4}\frac{\tilde{\phi}}{t^{3/2}}\right) \\ L_1^2\psi = \frac{1}{8i}f''(t)^{-2}\left(g_t^{(4)}(s)\psi(s) + 4g_t^{(3)}(t)\psi'(t)\right) = \frac{1}{4\pi}\left(3\left(\frac{\tilde{\phi}(t)}{\sqrt{t}}\right)' - 3\frac{\tilde{\phi}(t)}{t^{3/2}}\right) = \frac{1}{4\pi}\left(3\frac{\tilde{\phi}'(t)}{\sqrt{t}} - \frac{9}{2}\frac{\tilde{\phi}(t)}{t^{3/2}}\right) \\ L_1^3\psi = \frac{-1}{2^3 2! 3! i}f''(t)^{-3}(g_t^2)^{(6)}(t)\psi(s) = \frac{1}{4\pi}\left(\frac{15}{4}\frac{\tilde{\phi}(t)}{t^{3/2}}\right), \end{cases}$$

where $\tilde{\phi}(\tau) = \phi(\tau)/\tau$. Then, we have

$$
\begin{aligned}
L^1\psi &= L_1^1\psi + L_1^2\psi + L_1^3\psi \\
&= \frac{1}{4\pi}\left(\sqrt{t}\tilde{\phi}''(t) + (3-1)\frac{\tilde{\phi}'(t)}{\sqrt{t}} + \left(\frac{3}{4} - \frac{9}{2} + \frac{15}{4}\right)\frac{\tilde{\phi}(t)}{t^{3/2}}\right) = \frac{1}{4\pi\sqrt{t}}\left(t\tilde{\phi}(t)\right)'' = \frac{1}{4\pi\sqrt{t}}\phi''(t),
\end{aligned}
$$

and again Theorem A.1 shows that

$$
\left|\tilde{L}\phi(t) - \left(\phi(t) + \frac{\nu_s}{2c_s^2}t\phi''(t)\right)\right| \leq Ct\epsilon^{3/2}\sum_{\alpha\leq 4}\sup|\psi^{(\alpha)}(t)|.
$$

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